

# Quantum fields with topological defects

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Domain walls, strings and monopoles are extended objects, or defects, of quantum origin with topologically non-trivial properties and macroscopic behavior. They are described in Quantum Field Theory in terms of inhomogeneous condensates. We review the related formalism in the framework of the spontaneous breakdown of symmetry.

## I. INTRODUCTION

The ordered patterns we observe in condensed matter and in high energy physics are created by the quantum dynamics. Macroscopic systems exhibiting some kind of ordering, such as superconductors, ferromagnets, crystals, are described by the underlying quantum dynamics. Even the large scale structures in the Universe, as well as the ordering in the biological systems appear to be the manifestation of the microscopic dynamics ruling the elementary components of these systems. Thus we talk of *macroscopic quantum systems*: these are quantum systems in the sense that, although they behave classically, nevertheless some of their macroscopic features cannot be understood without recourse to quantum theory.

The question then arises of how the quantum dynamics generates the observed macroscopic properties. In other words, how it happens that the *macroscopic* scale characterizing those systems is dynamically generated out of the *microscopic* scale of the quantum elementary components [1].

Moreover, we also observe a variety of phenomena where quantum particles coexist and interact with *extended macroscopic objects* which show a classical behavior, e.g. vortices in superconductors and superfluids, magnetic domains in ferromagnets, dislocations and other *topological defects* (grain boundaries, point defects, etc.) in crystals, and so on.

We are thus faced also with the question of the quantum origin of topological defects and of their interaction with quanta [1]: This is a crucial issue for the understanding of symmetry breaking phase transitions and structure formation in a wide range of systems from condensed matter to cosmology [2–4].

Here, we will review how the generation of ordered structures and of extended objects is explained in Quantum Field Theory (QFT). We follow refs. [1] in our presentation. We will consider systems in which spontaneous symmetry breaking (SSB) occurs and show that topological defects originate by inhomogeneous (localized) condensation of quanta. The approach followed here is alternative to the usual one [5], in which one starts from the classical soliton solutions and then “quantizes” them, as well as to the QFT method based on dual (disorder) fields [6].

In Section 2 we first introduce some general features of QFT useful for our discussion and we then treat some aspects of SSB and the rearrangement of symmetry. In Section 3 we discuss the boson transformation theorem and the topological singularities of the boson condensate. Section 4 contains as an example a model with  $U(1)$  gauge invariance in which SSB, rearrangement of symmetry and topological defects are present [7]. There we show how macroscopic fields and currents are obtained from the microscopic quantum dynamics. The Nielsen-Olesen vortex solution is explicitly obtained as an example. Section 5 is devoted to conclusions.

## II. SYMMETRY AND ORDER IN QFT: A DYNAMICAL PROBLEM

QFT deals with systems with infinitely many degrees of freedom. The fields used for their description are *operator fields* whose mathematical significance is fully specified only when the state space where they operate is also assigned. This is the space of the states, or physical phase, of the system under given boundary conditions. A change in the boundary conditions may result in the transition of the system from one phase to another one. For example, a change of the temperature from above to below the critical temperature may induce the transition from the normal to the superconducting phase in a metal. The identification of the state space where the field operators have to be realized is thus a physically non trivial problem in QFT. In this respect, the QFT structure is drastically different from the one of Quantum Mechanics (QM). The reason is the following.

The von Neumann theorem in QM [8] states that for systems with a finite number of degrees of freedom all the irreducible representations of the canonical commutation relations are unitarily equivalent. Therefore in QM the physical system can only live in one single physical phase: unitary equivalence means indeed physical equivalence and

thus there is no room (no representations) for physically different phases. Such a situation drastically changes in QFT where systems with infinitely many degrees of freedom are treated. In such a case the von Neumann theorem does not hold and infinitely many unitarily inequivalent representations of the canonical commutation relations do in fact exist [9,10]. It is such a richness of QFT which allows the description of different physical phases.

### A. QFT as a two-level theory

In the perturbative approach, any quantum experiment or observation can be schematized as a scattering process where one prepares a set of free (non-interacting) particles (incoming particles or in-fields) which are then made to collide at some later time in some space region (space-time region of interaction). The products of the collision are expected to emerge out of the interaction region as free particles (outgoing particles or out-fields). Correspondingly, one has the in-field and the out-field state space. The interaction region is where the dynamics operates: given the in-fields and the in-states, the dynamics determines the out-fields and the out-states.

The incoming particles and the outgoing ones (also called quasi-particles in solid state physics) are well distinguishable and localizable particles only far away from the interaction region, at a time much before ( $t = -\infty$ ) and much after ( $t = +\infty$ ) the interaction time: in- and out-fields are thus said to be asymptotic fields, and for them the interaction forces are assumed not to operate (switched off).

The only regions accessible to observations are those far away (in space and in time) from the interaction region, i.e. the asymptotic regions (the in- and out-regions). It is so since, at the quantum level, observations performed in the interaction region or vacuum fluctuations there occurring may drastically interfere with the interacting objects thus changing their nature. Besides the asymptotic fields, one then also introduces dynamical or Heisenberg fields, i.e. the fields in terms of which the dynamics is given. Since the interaction region is precluded from observation, we do not observe Heisenberg fields. Observables are thus solely described in terms of asymptotic fields.

Summing up, QFT is a “two-level” theory: one level is the interaction level where the dynamics is specified by assigning the equations for the Heisenberg fields. The other level is the physical level, the one of the asymptotic fields and of the physical state space directly accessible to observations. The equations for the physical fields are equations for free fields, describing the observed incoming/outgoing particles.

To be specific, let the Heisenberg operator fields be generically denoted by  $\psi_H(x)$  and the physical operator fields by  $\varphi_{in}(x)$ <sup>1</sup>. They are both assumed to satisfy equal-time canonical (anti-)commutation relations.

For shortness, we omit considerations on the renormalization procedure, which are not essential for the conclusions we will reach. The Heisenberg field equations and the free field equations are written as

$$\Lambda(\partial) \psi_H(x) = \mathcal{J}[\psi_H](x) \quad (1)$$

$$\Lambda(\partial) \varphi_{in}(x) = 0, \quad (2)$$

where  $\Lambda(\partial)$  is a differential operator,  $x \equiv (t, \mathbf{x})$  and  $\mathcal{J}$  is some functional of the  $\psi_H$  fields, describing the interaction.

Eq.(1) can be formally recast in the following integral form (Yang–Feldman equation):

$$\psi_H(x) = \varphi_{in}(x) + \Lambda^{-1}(\partial) * \mathcal{J}[\psi_H](x), \quad (3)$$

where  $*$  denotes convolution. The symbol  $\Lambda^{-1}(\partial)$  denotes formally the Green function for  $\varphi_{in}(x)$ . The precise form of Green’s function is specified by the boundary conditions. Eq.(3) can be solved by iteration, thus giving an expression for the Heisenberg fields  $\psi_H(x)$  in terms of powers of the  $\varphi_{in}(x)$  fields; this is the Haag expansion in the LSZ formalism [9,11] (or “dynamical map” in the language of refs. [1]), which might be formally written as<sup>2</sup>

$$\psi_H(x) = F[x; \varphi_{in}]. \quad (4)$$

We stress that the equality in the dynamical map (4) is a “weak” equality, which means that it must be understood as an equality among matrix elements computed in the Hilbert space of the physical particles.

We observe that mathematical consistency in the above procedure requires that the set of  $\varphi_{in}$  fields must be an irreducible set; however, it may happen that not all the elements of the set are known since the beginning. For

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<sup>1</sup>For definiteness, we choose to work with the in-fields, although the set of out-fields would work equally well.

<sup>2</sup>A (formal) closed form for the dynamical map is obtained in the closed time path (CTP) formalism [12]. Then the Haag expansion (4) is directly applicable to both equilibrium and non-equilibrium situations.

example there might be composite (bound states) fields or even elementary quanta whose existence is ignored in a first recognition. Then the computation of the matrix elements in physical states will lead to the detection of unexpected poles in the Green's functions, which signal the existence of the ignored quanta. One thus introduces the fields corresponding to these quanta and repeats the computation. This way of proceeding is called the self-consistent method [1]. We remark that it is not necessary to have a one-to-one correspondence between the sets  $\{\psi_H^j\}$  and  $\{\varphi_{in}^i\}$ , as it happens whenever the set  $\{\varphi_{in}^i\}$  includes composite particles.

## B. The dynamical rearrangement of symmetry

As already mentioned, in QFT the Fock space for the physical states is not unique since one may have several physical phases, e.g. for a metal the normal phase and the superconducting phase, and so on. Fock spaces describing different phases are unitarily inequivalent spaces and correspondingly we have different expectation values for certain observables and even different irreducible sets of physical quanta. Thus, finding the dynamical map involves singling out the Fock space where the dynamics has to be realized.

Let us now suppose that the Heisenberg field equations are invariant under some group  $G$  of transformations of  $\psi_H$ :

$$\psi_H(x) \rightarrow \psi'_H(x) = g[\psi_H(x)] , \quad (5)$$

with  $g \in G$ . The symmetry is spontaneously broken when the vacuum state in the Fock space  $\mathcal{H}$  is not invariant under the group  $G$  but only under one of its subgroups [1,9,11].

On the other hand, Eq.(4) implies that when  $\psi_H$  is transformed as in (5), then

$$\varphi_{in}(x) \rightarrow \varphi'_{in}(x) = g'[\varphi_{in}(x)] , \quad (6)$$

with  $g'$  belonging to some group of transformations  $G'$  and such that

$$g[\psi_H(x)] = F[g'[\varphi_{in}(x)]] . \quad (7)$$

When symmetry is spontaneously broken it is  $G' \neq G$ , with  $G'$  the group contraction of  $G$  [13]; when symmetry is not broken  $G' = G$ .

Since  $G$  is the invariance group of the dynamics, Eq.(4) requires that  $G'$  is the group under which free fields equations are invariant, i.e. also  $\varphi'_{in}$  is a solution of (2). Since Eq.(4) is a weak equality,  $G'$  depends on the choice of the Fock space  $\mathcal{H}$  among the physically realizable unitarily inequivalent state spaces. Thus we see that the (same) original invariance of the dynamics may manifest itself in different symmetry groups for the  $\varphi_{in}$  fields according to different choices of the physical state space. Since this process is constrained by the dynamical equations (1), it is called the *dynamical rearrangement of symmetry* [1].

In conclusion, different ordering patterns appear to be different *manifestations* of the same basic dynamical invariance. The discovery of the process of the *dynamical rearrangement of symmetry* leads to a unified understanding of the dynamical generation of many observable ordered patterns. This is the phenomenon of the *dynamical generation of order*. The contraction of the symmetry group is the mathematical structure controlling the dynamical rearrangement of the symmetry [13]. For a qualitative presentation see Ref. [14].

One can now ask which ones are the carriers of the ordering information among the system elementary constituents and how the long range correlations and the coherence observed in ordered patterns are generated and sustained. The answer is in the fact that SSB implies the appearance of boson particles [15,16], the so called Nambu-Goldstone (NG) modes or quanta. They manifest as long range correlations and thus they are responsible of the above mentioned change of scale, from microscopic to macroscopic. The coherent boson condensation of NG modes turns out to be the mechanism by which order is generated, as we will see in an explicit example in Section 4.

## III. THE “BOSON TRANSFORMATION” METHOD

We now discuss the quantum origin of extended objects (defects) and show how they naturally emerge as macroscopic objects (inhomogeneous condensates) from the quantum dynamics. At zero temperature, the classical soliton solutions are then recovered in the Born approximation. This approach is known as the “boson transformation” method [1].

## A. The boson transformation theorem

Let us consider, for simplicity, the case of a dynamical model involving one scalar field  $\psi_H$  and one asymptotic field  $\varphi_{in}$  satisfying Eqs.(1) and (2), respectively.

As already remarked, the dynamical map is valid only in a weak sense, i.e. as a relation among matrix elements. This implies that Eq.(4) is not unique, since different sets of asymptotic fields and the corresponding Hilbert spaces can be used in its construction. Let us indeed consider a c-number function  $f(x)$ , satisfying the  $\varphi_{in}$  equations of motion (2):

$$\Lambda(\partial) f(x) = 0. \quad (8)$$

The *boson transformation theorem* [1] states that the field

$$\psi_H^f(x) = F[x; \varphi_{in} + f]. \quad (9)$$

is also a solution of the Heisenberg equation (1). The corresponding Yang–Feldman equation takes the form

$$\psi_H^f(x) = \varphi_{in}(x) + f(x) + \Lambda^{-1}(\partial) * \mathcal{J}[\psi_H^f](x). \quad (10)$$

The difference between the two solutions  $\psi_H$  and  $\psi_H^f$  is only in the boundary conditions. An important point is that the expansion Eq.(9) is obtained from that in Eq.(4), by the space–time dependent translation

$$\varphi_{in}(x) \rightarrow \varphi_{in}(x) + f(x). \quad (11)$$

The essence of the boson transformation theorem is that the dynamics embodied in Eq.(1), contains an internal freedom, represented by the possible choices of the function  $f(x)$ , satisfying the free field equation (8).

We also observe that the transformation (11) is a canonical transformation since it leaves invariant the canonical form of commutation relations.

Let  $|0\rangle$  denote the vacuum for the free field  $\varphi_{in}$ . The vacuum expectation value of Eq.(10) gives:

$$\phi^f(x) \equiv \langle 0 | \psi_H^f(x) | 0 \rangle = f(x) + \langle 0 | [\Lambda^{-1}(\partial) * \mathcal{J}[\psi_H^f](x)] | 0 \rangle. \quad (12)$$

The c-number field  $\phi^f(x)$  is the order parameter. We remark that it is fully determined by the quantum dynamics. In the classical or Born approximation, which consists in taking  $\langle 0 | \mathcal{J}[\psi_H^f] | 0 \rangle = \mathcal{J}[\phi^f]$ , i.e. neglecting all the contractions of the physical fields, we define  $\phi_{cl}^f(x) \equiv \lim_{\hbar \rightarrow 0} \phi^f(x)$ . In this limit we have

$$\Lambda(\partial) \phi_{cl}^f(x) = \mathcal{J}[\phi_{cl}^f](x), \quad (13)$$

i.e.  $\phi_{cl}^f(x)$  provides the solution of the classical Euler–Lagrange equation.

Beyond the classical level, in general, the form of this equation changes. The Yang–Feldman equation (10) gives not only the equations for the order parameter Eq.(13), but also, at higher orders in  $\hbar$ , the dynamics of the physical quanta in the potential generated by the “macroscopic object”  $\phi^f(x)$  [1].

One can show [1], that the class of solutions of Eq.(8) which lead to topologically non-trivial (i.e. carrying a non-zero topological charge) solutions of Eq.(13), are those which have some sort of singularity with respect to Fourier transform. These can be either *divergent singularities* or *topological singularities*. The first are associated to a divergence of  $f(x)$  for  $|\mathbf{x}| = \infty$ , at least in some direction. Topological singularities are instead present when  $f(x)$  is not single-valued, i.e. it is path dependent. In both cases, the macroscopic object described by the order parameter, carries a non-zero topological charge.

## B. Topological singularities and massless bosons

An important result is that the boson transformation functions carrying topological singularities are only allowed for massless bosons [1].

Consider a generic boson field  $\chi_{in}$  satisfying the equation

$$(\partial^2 + m^2)\chi_{in}(x) = 0, \quad (14)$$

and suppose that the function  $f(x)$  for the boson transformation  $\chi_{in}(x) \rightarrow \chi_{in}(x) + f(x)$  carries a topological singularity. It is then not single-valued and thus path-dependent:

$$G_{\mu\nu}^+(x) \equiv [\partial_\mu, \partial_\nu] f(x) \neq 0, \quad \text{for certain } \mu, \nu, x. \quad (15)$$

On the other hand,  $\partial_\mu f(x)$ , which is related with observables, is single-valued, i.e.  $[\partial_\rho, \partial_\nu] \partial_\mu f(x) = 0$ . Recall that  $f(x)$  is solution of the  $\chi_{in}$  equation:

$$(\partial^2 + m^2)f(x) = 0. \quad (16)$$

From the definition of  $G_{\mu\nu}^+(x)$  and the regularity of  $\partial_\mu f(x)$  it follows, by computing  $\partial^\mu G_{\mu\nu}^+(x)$ , that

$$\partial_\mu f(x) = \frac{1}{\partial^2 + m^2} \partial^\lambda G_{\lambda\mu}^+(x). \quad (17)$$

This equation and the antisymmetric nature of  $G_{\mu\nu}^+(x)$  then lead to  $\partial^2 f(x) = 0$ , which in turn implies  $m = 0$ . Thus we conclude that (15) is only compatible with massless equation for  $\chi_{in}$ .

The topological charge is defined as

$$N_T = \int_C dl^\mu \partial_\mu f = \int_S dS_\mu \epsilon^{\mu\nu\sigma} \partial_\nu \partial_\sigma f = \frac{1}{2} \int_S dS^{\mu\nu} G_{\mu\nu}^+. \quad (18)$$

Here  $C$  is a contour enclosing the singularity and  $S$  a surface with  $C$  as boundary.  $N_T$  does not depend on the path  $C$  provided this does not cross the singularity. The dual tensor  $G^{\mu\nu}(x)$  is

$$G^{\mu\nu}(x) \equiv -\frac{1}{2} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho}^+(x) \quad (19)$$

and satisfies the continuity equation:

$$\partial_\mu G^{\mu\nu}(x) = 0 \quad \Leftrightarrow \quad \partial_\mu G_{\lambda\rho}^+(x) + \partial_\rho G_{\mu\lambda}^+(x) + \partial_\lambda G_{\rho\mu}^+(x) = 0. \quad (20)$$

Eq.(20) completely characterizes the topological singularity [1].

#### IV. AN EXAMPLE: THE ANDERSON-HIGGS-KIBBLE MECHANISM AND THE VORTEX SOLUTION

We consider a model of a complex scalar field  $\phi(x)$  interacting with a gauge field  $A_\mu(x)$  [17–19]. The lagrangian density  $\mathcal{L}[\phi(x), \phi^*(x), A_\mu(x)]$  is invariant under the global and the local  $U(1)$  gauge transformations<sup>3</sup>:

$$\phi(x) \rightarrow e^{i\theta} \phi(x), \quad A_\mu(x) \rightarrow A_\mu(x), \quad (21)$$

$$\phi(x) \rightarrow e^{ie_0\lambda(x)} \phi(x), \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x), \quad (22)$$

respectively, where  $\lambda(x) \rightarrow 0$  for  $|x_0| \rightarrow \infty$  and/or  $|\mathbf{x}| \rightarrow \infty$  and  $e_0$  is the coupling constant. We work in the Lorentz gauge  $\partial_\mu A^\mu(x) = 0$ . The generating functional, including the gauge constraint, is [7]

$$\mathcal{Z}[J, K] = \frac{1}{\mathcal{N}} \int [dA_\mu][d\phi][d\phi^*][dB] \exp[i\mathcal{S}[A_\mu, B, \phi]], \quad (23)$$

$$\mathcal{S} = \int d^4x [\mathcal{L}(x) + B(x)\partial^\mu A_\mu(x) + K^*(x)\phi(x) + K(x)\phi^*(x) + J^\mu(x)A_\mu(x) + i\epsilon|\phi(x) - v|^2],$$

$$\mathcal{N} = \int [dA_\mu][d\phi][d\phi^*][dB] \exp\left[i \int d^4x (\mathcal{L}(x) + i\epsilon|\phi(x) - v|^2)\right].$$

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<sup>3</sup>We do not assume a particular form for the Lagrangian density, so the following results are quite general.

$B(x)$  is an auxiliary field which implements the gauge fixing condition [7,20]. Notice the  $\epsilon$ -term where  $v$  is a complex number. Its rôle is to specify the condition of symmetry breaking under which we want to compute the functional integral and it may be given the physical meaning of a small external field triggering the symmetry breaking [7]. The limit  $\epsilon \rightarrow 0$  must be made at the end of the computations. We will use the notation

$$\langle F[\phi] \rangle_{\epsilon, J, K} \equiv \frac{1}{\mathcal{N}} \int [dA_\mu][d\phi][d\phi^*][dB] F[\phi] \exp[i\mathcal{S}[A_\mu, B, \phi]] , \quad (24)$$

with  $\langle F[\phi] \rangle_\epsilon \equiv \langle F[\phi] \rangle_{\epsilon, J=K=0}$  and  $\langle F[\phi] \rangle \equiv \lim_{\epsilon \rightarrow 0} \langle F[\phi] \rangle_\epsilon$ .

The fields  $\phi$ ,  $A_\mu$  and  $B$  appearing in the generating functional are c-number fields. In the following the Heisenberg operator fields corresponding to them will be denoted by  $\phi_H$ ,  $A_{H\mu}$  and  $B_H$ , respectively. Thus the spontaneous symmetry breaking condition is expressed by  $\langle 0|\phi_H(x)|0 \rangle \equiv \tilde{v} \neq 0$ , with  $\tilde{v}$  constant.

Since in the functional integral formalism the functional average of a given c-number field gives the vacuum expectation value of the corresponding operator field, e.g.  $\langle F[\phi] \rangle \equiv \langle 0|F[\phi_H]|0 \rangle$ , we have  $\lim_{\epsilon \rightarrow 0} \langle \phi(x) \rangle_\epsilon \equiv \langle 0|\phi_H(x)|0 \rangle = \tilde{v}$ .

Let us introduce the following decompositions:  $\phi(x) = \frac{1}{\sqrt{2}}[\psi(x) + i\chi(x)]$ ,  $K(x) = \frac{1}{\sqrt{2}}[K_1(x) + iK_2(x)]$  and  $\rho(x) \equiv \psi(x) - \langle \psi(x) \rangle_\epsilon$ . Note that  $\langle \chi(x) \rangle_\epsilon = 0$  because of the invariance under  $\chi \rightarrow -\chi$ .

### A. The Goldstone theorem

Since the functional integral (23) is invariant under the global transformation (21), we have that  $\partial \mathcal{Z}[J, K]/\partial \theta = 0$  and subsequent derivatives with respect to  $K_1$  and  $K_2$  lead to

$$\langle \psi(x) \rangle_\epsilon = \sqrt{2}\epsilon v \int d^4y \langle \chi(x)\chi(y) \rangle_\epsilon = \sqrt{2}\epsilon v \Delta_\chi(\epsilon, 0) . \quad (25)$$

In momentum space the propagator for the field  $\chi$  has the general form

$$\Delta_\chi(0, p) = \lim_{\epsilon \rightarrow 0} \left[ \frac{Z_\chi}{p^2 - m_\chi^2 + i\epsilon a_\chi} + (\text{continuum contributions}) \right] . \quad (26)$$

Here  $Z_\chi$  and  $a_\chi$  are renormalization constants. The integration in Eq.(25) picks up the pole contribution at  $p^2 = 0$ , and leads to

$$\tilde{v} = \sqrt{2} \frac{Z_\chi}{a_\chi} v \Leftrightarrow m_\chi = 0 \quad ; \quad \tilde{v} = 0 \Leftrightarrow m_\chi \neq 0 . \quad (27)$$

The Goldstone theorem [15] is thus proved: if the symmetry is spontaneously broken ( $\tilde{v} \neq 0$ ), a massless mode must exist, whose field is  $\chi(x)$ , i.e. the NG boson mode. Since it is massless it manifests as a long range correlation mode. (Notice that in the present case of a complex scalar field model the NG mode is an elementary field. In other models it may appear as a bound state, e.g. the magnon in (anti-)ferromagnets). Note that

$$\frac{\partial}{\partial v} \langle \psi(x) \rangle_\epsilon = \sqrt{2}\epsilon \int d^4y \langle \rho(x)\rho(y) \rangle_\epsilon , \quad (28)$$

and because  $m_\rho \neq 0$ , the r.h.s. of this equation vanishes in the limit  $\epsilon \rightarrow 0$ ; therefore  $\tilde{v}$  is independent of  $|v|$ , although the phase of  $|v|$  determines the one of  $\tilde{v}$  (from Eq.(25)): as in ferromagnets, once an external magnetic field is switched on, the system is magnetized independently of the strength of the external field.

### B. The dynamical map and the field equations

Observing that the change of variables (21) (and/or (22)) does not affect the generating functional, we may obtain the Ward-Takahashi identities. Also, using  $B(x) \rightarrow B(x) + \lambda(x)$  in (23) gives  $\langle \partial^\mu A_\mu(x) \rangle_{\epsilon, J, K} = 0$ . One then finds the following two-point function pole structures [7]:

$$\langle B(x)\chi(y) \rangle = \lim_{\epsilon \rightarrow 0} \left\{ \frac{-i}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \frac{e_0 \tilde{v}}{p^2 + i\epsilon a_\chi} \right\} , \quad (29)$$

$$\langle B(x)A^\mu(y) \rangle = \partial_x^\mu \frac{i}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \frac{1}{p^2} , \quad (30)$$

$$\langle B(x)B(y) \rangle = \lim_{\epsilon \rightarrow 0} \left\{ \frac{-i}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \frac{(e_0 \tilde{v})^2}{Z_\chi} \left[ \frac{1}{p^2 + i\epsilon a_\chi} - \frac{1}{p^2} \right] \right\} . \quad (31)$$

The absence of branch cut singularities in propagators (29)–(31) suggests that  $B(x)$  obeys a free field equation. In addition, Eq.(31) indicates that the model contains a massless negative norm state (ghost) besides the NG massless mode  $\chi$ . Moreover, it can be shown [7] that a massive vector field  $U_{in}^\mu$  also exists in the theory. Note that because of the invariance  $(\chi, A_\mu, B) \rightarrow (-\chi, -A_\mu, -B)$ , all the other two-point functions must vanish.

The dynamical maps expressing the Heisenberg operator fields in terms of the asymptotic operator fields, are found to be [7]:

$$\phi_H(x) =: \exp \left\{ i \frac{Z_\chi^{\frac{1}{2}}}{\tilde{v}} \chi_{in}(x) \right\} \left[ \tilde{v} + Z_\rho^{\frac{1}{2}} \rho_{in}(x) + \mathcal{F}[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})] \right] :, \quad (32)$$

$$A_H^\mu(x) = Z_3^{\frac{1}{2}} U_{in}^\mu(x) + \frac{Z_\chi^{\frac{1}{2}}}{e_0 \tilde{v}} \partial^\mu b_{in}(x) + : \mathcal{F}^\mu[\rho_{in}, U_{in}^\mu, \partial(\chi_{in} - b_{in})] :, \quad (33)$$

$$B_H(x) = \frac{e_0 \tilde{v}}{Z_\chi^{\frac{1}{2}}} [b_{in}(x) - \chi_{in}(x)] + c, \quad (34)$$

where  $: \dots :$  denotes the normal ordering and the functionals  $\mathcal{F}$  and  $\mathcal{F}^\mu$  are to be determined within a particular model. In Eqs.(32)–(34),  $\chi_{in}$  denotes the NG mode,  $b_{in}$  the ghost mode,  $U_{in}^\mu$  the massive vector field and  $\rho_{in}$  the massive matter field. In Eq.(34)  $c$  is a c-number constant, whose value is irrelevant since only derivatives of  $B$  appear in the field equations (see below).  $Z_3$  represents the wave function renormalization for  $U_{in}^\mu$ . The corresponding field equations are

$$\partial^2 \chi_{in}(x) = 0, \quad \partial^2 b_{in}(x) = 0, \quad (\partial^2 + m_\rho^2) \rho_{in}(x) = 0, \quad (35)$$

$$(\partial^2 + m_V^2) U_{in}^\mu(x) = 0, \quad \partial_\mu U_{in}^\mu(x) = 0. \quad (36)$$

with  $m_V^2 = \frac{Z_3}{Z_\chi} (e_0 \tilde{v})^2$ . The field equations for  $B_H$  and  $A_{H\mu}$  read [7]

$$\partial^2 B_H(x) = 0, \quad -\partial^2 A_{H\mu}(x) = j_{H\mu}(x) - \partial_\mu B_H(x), \quad (37)$$

with  $j_{H\mu}(x) = \delta \mathcal{L}(x) / \delta A_H^\mu(x)$ . One may then require that the current  $j_{H\mu}$  is the only source of the gauge field  $A_{H\mu}$  in any observable process. This amounts to impose the condition:  ${}_p \langle b | \partial_\mu B_H(x) | a \rangle_p = 0$ , i.e.

$$(-\partial^2) {}_p \langle b | A_{H\mu}^0(x) | a \rangle_p = {}_p \langle b | j_{H\mu}(x) | a \rangle_p, \quad (38)$$

where  $|a\rangle_p$  and  $|b\rangle_p$  denote two generic *physical* states and  $A_{H\mu}^{0\mu}(x) \equiv A_H^\mu(x) - e_0 \tilde{v} : \partial^\mu b_{in}(x) :$ . Eq.(38) are the classical Maxwell equations. The condition  ${}_p \langle b | \partial_\mu B_H(x) | a \rangle_p = 0$  leads to the Gupta–Bleuler–like condition

$$[\chi_{in}^{(-)}(x) - b_{in}^{(-)}(x)] | a \rangle_p = 0, \quad (39)$$

where  $\chi_{in}^{(-)}$  and  $b_{in}^{(-)}$  are the positive-frequency parts of the corresponding fields. Thus we see that  $\chi_{in}$  and  $b_{in}$  cannot participate in any observable reaction. This is confirmed by the fact that they are present in the  $S$  matrix in the combination  $(\chi_{in} - b_{in})$  [7]. It is to be remarked however that the NG boson does not disappear from the theory: we shall see below that there are situations in which the NG fields do have observable effects.

### C. The dynamical rearrangement of symmetry and the classical fields and currents

From Eqs.(32)–(33) we see that the local gauge transformations of the Heisenberg fields

$$\phi_H(x) \rightarrow e^{ie_0 \lambda(x)} \phi_H(x), \quad A_H^\mu(x) \rightarrow A_H^\mu(x) + \partial^\mu \lambda(x), \quad B_H(x) \rightarrow B_H(x), \quad (40)$$

with  $\partial^2 \lambda(x) = 0$ , are induced by the in-field transformations

$$\begin{aligned} \chi_{in}(x) &\rightarrow \chi_{in}(x) + \frac{e_0 \tilde{v}}{Z_\chi^{\frac{1}{2}}} \lambda(x), & b_{in}(x) &\rightarrow b_{in}(x) + \frac{e_0 \tilde{v}}{Z_\chi^{\frac{1}{2}}} \lambda(x), \\ \rho_{in}(x) &\rightarrow \rho_{in}(x), & U_{in}^\mu(x) &\rightarrow U_{in}^\mu(x). \end{aligned} \quad (41)$$

On the other hand, the global phase transformation  $\phi_H(x) \rightarrow e^{i\theta} \phi_H(x)$  is induced by

$$\begin{aligned} \chi_{in}(x) &\rightarrow \chi_{in}(x) + \frac{\tilde{v}}{Z_\chi^{\frac{1}{2}}} \theta f(x) \quad , \quad b_{in}(x) \rightarrow b_{in}(x) \quad , \\ \rho_{in}(x) &\rightarrow \rho_{in}(x) \quad , \quad U_{in}^\mu(x) \rightarrow U_{in}^\mu(x) \quad , \end{aligned} \quad (42)$$

with  $\partial^2 f(x) = 0$  and the limit  $f(x) \rightarrow 1$  to be performed at the end of computations. Note that under the above transformations the in-field equations and the  $S$  matrix are invariant and that  $B_H$  is changed by an irrelevant c-number (in the limit  $f \rightarrow 1$ ).

Consider now the boson transformation  $\chi_{in}(x) \rightarrow \chi_{in}(x) + \alpha(x)$ : In local gauge theories the boson transformation must be compatible with the Heisenberg field equations but also with the physical state condition (39). Under the boson transformation with  $\alpha(x) = \tilde{v} Z_\chi^{-\frac{1}{2}} \theta f(x)$  and  $\partial^2 f(x) = 0$ ,  $B_H$  changes as

$$B_H(x) \rightarrow B_H(x) - \frac{e_0 \tilde{v}^2}{Z_\chi} f(x) \quad , \quad (43)$$

Eq.(38) is thus violated when the Gupta-Bleuler-like condition is imposed. In order to restore it, the shift in  $B_H$  must be compensated by means of the transformation on  $U_{in}^\mu$ :

$$U_{in}^\mu(x) \rightarrow U_{in}^\mu(x) + Z_3^{-\frac{1}{2}} a^\mu(x) \quad , \quad \partial_\mu a^\mu(x) = 0 \quad , \quad (44)$$

with a convenient c-number function  $a^\mu(x)$ . The dynamical maps of the various Heisenberg operators are not affected by (44) since they contain  $U_{in}^\mu$  and  $B_H$  in a combination such that the changes of  $B_H$  and of  $U_{in}^\mu$  compensate each other provided

$$(\partial^2 + m_V^2) a_\mu(x) = \frac{m_V^2}{e_0} \partial_\mu f(x) \quad . \quad (45)$$

Eq. (45) thus obtained is the Maxwell equation for the massive potential vector  $a_\mu$  [7]. The classical ground state current  $j^\mu$  turns out to be

$$j^\mu(x) \equiv \langle 0 | j_H^\mu(x) | 0 \rangle = m_V^2 \left[ a^\mu(x) - \frac{1}{e_0} \partial^\mu f(x) \right] \quad . \quad (46)$$

The term  $m_V^2 a^\mu(x)$  is the *Meissner current*, while  $\frac{m_V^2}{e_0} \partial^\mu f(x)$  is the *boson current*. The key point here is that both the macroscopic field and current are given in terms of the boson condensation function  $f(x)$ .

Two remarks are in order: First, note that the terms proportional to  $\partial^\mu f(x)$  are related to observable effects, e.g. the boson current which acts as the source of the classical field. Second, note that the macroscopic ground state effects do not occur for regular  $f(x)$  ( $G_{\mu\nu}^+(x) = 0$ ). In fact, from (45) we obtain  $a_\mu(x) = \frac{1}{e_0} \partial_\mu f(x)$  for regular  $f(x)$  which implies zero classical current ( $j_\mu = 0$ ) and zero classical field ( $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ ), since the Meissner and the boson current cancel each other.

In conclusion, the vacuum current appears only when  $f(x)$  has topological singularities and these can be created only by condensation of massless bosons, i.e. when SSB occurs. This explains why topological defects appear in the process of phase transitions, where NG modes are present and gradients in their condensate densities are nonzero [2,3].

On the other hand, the appearance of space-time order parameter is no guarantee that persistent ground state currents (and fields) will exist: if  $f(x)$  is a regular function, the space-time dependence of  $\tilde{v}$  can be gauged away by an appropriate gauge transformation.

Since, as said, the boson transformation with regular  $f(x)$  does not affect observable quantities, the  $S$  matrix is actually given by

$$S = : S[\rho_{in}, U_{in}^\mu - \frac{1}{m_V} \partial(\chi_{in} - b_{in})] : \quad . \quad (47)$$

This is indeed independent of the boson transformation with regular  $f(x)$ :

$$S \rightarrow S' = : S[\rho_{in}, U_{in}^\mu - \frac{1}{m_V} \partial(\chi_{in} - b_{in}) + Z_3^{-\frac{1}{2}} (a^\mu - \frac{1}{e_0} \partial^\mu f)] : \quad (48)$$

since  $a_\mu(x) = \frac{1}{e_0} \partial_\mu f(x)$  for regular  $f(x)$ . However,  $S' \neq S$  for singular  $f(x)$ :  $S'$  includes the interaction of the quanta  $U_{in}^\mu$  and  $\phi_{in}$  with the classically behaving macroscopic defects [1].



### D. The vortex solution

Below we consider the example of the Nielsen–Olesen vortex string solution. We show which one is the boson function  $f(x)$  controlling the non-homogeneous NG boson condensation in terms of which the string solution is described. For shortness, we only report the results of the computations. The detailed derivation as well as the discussion of further examples can be found in Ref. [1].

In the present  $U(1)$  problem, the electromagnetic tensor and the vacuum current are [1,7]

$$F_{\mu\nu}(x) = \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) = 2\pi \frac{m_V^2}{e_0} \int d^4x' \Delta_c(x-x') G_{\mu\nu}^+(x') , \quad (49)$$

$$j_\mu(x) = -2\pi \frac{m_V^2}{e_0} \int d^4x' \Delta_c(x-x') \partial_{x'}^\nu G_{\nu\mu}^+(x') , \quad (50)$$

respectively, and satisfy  $\partial^\mu F_{\mu\nu}(x) = -j_\nu(x)$ . In these equations

$$\Delta_c(x-x') = \frac{1}{(2\pi)^4} \int d^4p e^{-ip(x-x')} \frac{1}{p^2 - m_V^2 + i\epsilon} . \quad (51)$$

The line singularity for the vortex (or string) solution can be parameterized by a single line parameter  $\sigma$  and by the time parameter  $\tau$ . A *static vortex* solution is obtained by setting  $y_0(\tau, \sigma) = \tau$  and  $\mathbf{y}(\tau, \sigma) = \mathbf{y}(\sigma)$ , with  $y$  denoting the line coordinate.  $G_{\mu\nu}^+(x)$  is non-zero only on the line at  $y$  (we can consider more lines but let us limit to only one line, for simplicity). Thus, we have:

$$\begin{aligned} G_{0i}(x) &= \int d\sigma \frac{dy_i(\sigma)}{d\sigma} \delta^3[\mathbf{x} - \mathbf{y}(\sigma)] , & G_{ij}(x) &= 0 , \\ G_{ij}^+(x) &= -\epsilon_{ijk} G_{0k}(x) , & G_{0i}^+(x) &= 0 . \end{aligned} \quad (52)$$

Eq.(49) shows that these vortices are purely *magnetic*. We obtain

$$\partial_0 f(x) = 0 , \quad \partial_i f(x) = \frac{1}{(2\pi)^2} \int d\sigma \epsilon_{ijk} \frac{dy_k(\sigma)}{d\sigma} \partial_j^x \int d^3p \frac{e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y}(\sigma))}}{\mathbf{p}^2} , \quad (53)$$

i.e., by using the identity  $(2\pi)^{-2} \int d^3p \frac{e^{i\mathbf{p} \cdot \mathbf{x}}}{\mathbf{p}^2} = \frac{1}{2|\mathbf{x}|}$ ,

$$\nabla f(x) = -\frac{1}{2} \int d\sigma \frac{d\mathbf{y}_k(\sigma)}{d\sigma} \wedge \nabla_x \frac{1}{|\mathbf{x} - \mathbf{y}(\sigma)|} , \quad (54)$$

Note that  $\nabla^2 f(x) = 0$  is satisfied.

A straight infinitely long vortex is specified by  $y_i(\sigma) = \sigma \delta_{i3}$  with  $-\infty < \sigma < \infty$ . The only non vanishing component of  $G^{\mu\nu}(x)$  are  $G^{03}(x) = G_{12}^+(x) = \delta(x_1)\delta(x_2)$ . Eq.(54) gives [1,7]

$$\begin{aligned} \frac{\partial}{\partial x_1} f(x) &= \frac{1}{2} \int d\sigma \frac{\partial}{\partial x_2} [x_1^2 + x_2^2 + (x_3 - \sigma)^2]^{-\frac{1}{2}} = -\frac{x_2}{x_1^2 + x_2^2} , \\ \frac{\partial}{\partial x_2} f(x) &= \frac{x_1}{x_1^2 + x_2^2} , \quad \frac{\partial}{\partial x_3} f(x) = 0 , \end{aligned} \quad (55)$$

and then

$$f(x) = \tan^{-1} \left( \frac{x_2}{x_1} \right) = \theta(x) . \quad (56)$$

We have thus determined the boson transformation function corresponding to a particular vortex solution. The vector potential is

$$\begin{aligned} a_1(x) &= -\frac{m_V^2}{2e_0} \int d^4x' \Delta_c(x-x') \frac{x'_2}{x_1'^2 + x_2'^2} , \\ a_2(x) &= \frac{m_V^2}{2e_0} \int d^4x' \Delta_c(x-x') \frac{x'_1}{x_1'^2 + x_2'^2} , \\ a_3(x) &= a_0(x) = 0 , \end{aligned} \quad (57)$$

and the only non-vanishing component of  $F_{\mu\nu}$ :

$$F_{12}(x) = -2\pi \frac{m_V^2}{e_0} \int d^4x' \Delta_c(x-x') \delta(x'_1) \delta(x'_2) = \frac{m_V^2}{e_0} K_0 \left( m_V \sqrt{x_1^2 + x_2^2} \right). \quad (58)$$

Finally, the vacuum current Eq.(50) is given by

$$\begin{aligned} j_1(x) &= -\frac{m_V^3}{e_0} \frac{x_2}{\sqrt{x_1^2 + x_2^2}} K_1 \left( m_V \sqrt{x_1^2 + x_2^2} \right), \\ j_2(x) &= \frac{m_V^3}{e_0} \frac{x_1}{\sqrt{x_1^2 + x_2^2}} K_1 \left( m_V \sqrt{x_1^2 + x_2^2} \right), \\ j_3(x) &= j_0(x) = 0. \end{aligned} \quad (59)$$

We observe that these results are the same of the Nielsen-Olesen vortex solution [22]. Notice that we did not specify the potential in our model but only the invariance properties. Thus, the invariance properties of the dynamics determine the characteristics of the topological solutions. The vortex solution manifests the original  $U(1)$  symmetry through the cylindric angle  $\theta$  which is the parameter of the  $U(1)$  representation in the coordinate space.

## V. CONCLUSIONS

We have discussed how topological defects arise as inhomogeneous condensates in Quantum Field Theory. Topological defects are shown to have a genuine quantum nature. The approach reviewed here goes under the name of “boson transformation method” and relies on the existence of unitarily inequivalent representations of the field algebra in QFT.

Describing quantum fields with topological defects amounts then to properly choose the physical Fock space for representing the Heisenberg field operators. Once the boundary conditions corresponding to a particular soliton sector are found, then the Heisenberg field operators embodied with such conditions contain the full information about the defects, the quanta and their mutual interaction. One can thus calculate Green’s functions for particles in the presence of defects. The extension to finite temperature is discussed in Refs. [12,21].

As an example we have discussed a model with  $U(1)$  gauge invariance and SSB and we have obtained the Nielsen-Olesen vortex solution [22] in terms of localized condensation of Goldstone bosons. These thus appear to play a physical role, although, in the presence of gauge fields, they do not show up in the physical spectrum as excitation quanta. The function  $f(x)$  controlling the condensation of the NG bosons must be singular in order to produce observable effects. Boson transformations with regular  $f(x)$  only amount to gauge transformations. For the treatment of topological defects in non-abelian gauge theories, see Ref. [21].

Finally, when there are no NG modes, as in the case of the kink solution or the sine-Gordon solution, the boson transformation function has to carry divergence singularity at spatial infinity [1,12]. In ref. [23] the boson transformation has been also discussed in connection with the Bäcklund transformation at a classical level and the confinement of the constituent quanta in the coherent condensation domain.

For further reading on quantum fields with topological defects, see Ref. [24].

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